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COUNTABLE SPREAD OF exp Y AND λY

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We show that it is consistent with ZFC that there exists a compact 0-dimensional Hausdorff space X for which $\exp X$ has countable spread, but X is not metrizable. This establishes the independence of Malyhin's problem. The space X also has no uncountable weakly separated subspaces, its superextension is first countable, and its square is a strong S-space. For 0-dimensional Y we prove that λY has countable spread iff Y is compact and metrizable. We show that it is consistent with ZFC that if Y is 0-dimensional and λY is first countable, then Y is compact and metrizable.

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forcing	countable spread	hereditarily	exp Y
separable	weakly separated	first countable	λY

1. Introduction

In [3], it is shown that MA + $\neg$CH implies that if $\exp Y$ has countable spread, then Y is compact and metrizable. Herein, we show that it is consistent with ZFC that there exists a compact 0-dimensional Hausdorff space X for which $\exp X$ has countable spread but X is not metrizable. This establishes the independence of Malyhin's conjecture that $\exp Y$ has countable spread iff Y is compact and metrizable, see [7]. Our example X is very strong; among its many properties we show that X has no uncountable weakly separated subspaces.

In Section 3, we prove that if $\exp Y$ has countable spread, then $\exp Y$ is hereditarily separable and every finite power of Y is hereditarily separable. Thus, our preceding example X has $\exp X$ an S-space and X^2 a strong S-space while X is not an S-space.

In Section 4, we prove for 0-dimensional Y that λY has countable spread iff Y is compact and metrizable. It is proven that if exp Y has countable spread, then λY is first countable. Thus, our example X has λX first countable as well.

2. Preliminaries

Our set-theoretic and topological terminology are standard. All topological spaces are assumed to be regular and Hausdorff. When speaking of the extension spaces exp Y and λY, we further assume Y to be normal.

The first infinite ordinal is denoted by ω and the first uncountable ordinal by ω_1. If f is a function from A to B and $D \subseteq A$, then $f \upharpoonright D$ is the restriction of the function f to the subset D. If H and K are subsets of ω_1, then $H < K$ means that every ordinal in H is less than every ordinal in K. If $H = \{\alpha\}$, then we simply write $\alpha < K$.

Y has *countable spread*, if Y contains no uncountable discrete subspaces. Y is *weakly-separated*, see [13], if there exists a reflexive and antisymmetric relation R on Y such that for each $y \in Y$, $\{x \in Y: xRy\}$ is open in Y. Y is *left-separated*, if Y is weakly-separated by the inverse of a well-ordered relation. It is well-known that Y is hereditarily separable iff Y contains no uncountable left-separated subspaces. An *S-space* is a space which is hereditarily separable but not hereditarily Lindelöf. A recent result of S. Todorčević is the following. It is consistent with ZFC that there are no S-spaces.

If Y is a compact 0-dimensional space, then the Boolean algebra of clopen sets of Y is denoted by $\mathscr{B}(Y)$. A subset S of $\mathscr{B}(Y)$ is called a *strong antichain* if no element x of S is contained in the union of any finite subset of $S - \{x\}$. We will use the following result of [2]: Let Y be a compact 0-dimensional space and let κ be an infinite cardinal. Then Y contains a discrete subset of cardinality κ iff $\mathscr{B}(Y)$ contains a strong antichain of cardinality κ.

Exp Y denotes the set of all non-empty closed subsets of Y. If $G_1, \ldots, G_n$ are subsets of Y, we define

$$\langle G_1, \ldots, G_n \rangle = \{F \in \exp Y : F \subseteq \bigcup_{i=1}^{n} G_i \text{ and } F \cap G_i \neq \emptyset \text{ for } i = 1, \ldots, n\}.$$

In particular, if $G \subseteq Y$, then $\langle G \rangle = \{F \in \exp Y : F \subseteq G\}$. The Vietoris topology on exp Y has the sets of the form $\langle G_1, \ldots, G_n \rangle$ where $G_1, \ldots, G_n$ are open in Y, as an open basis. A comprehensive discussion of exp Y can be found in [8]. For $n < \omega$, the symmetric n-fold product $\langle Y \rangle^n$ is the subspace $\{F \in \exp Y : |F| \leq n\}$ of exp Y. One fact that we will need is that if H and K are disjoint closed subsets of Y, then $\exp H \times \exp K$ is embedded in exp Y by the union map.

A family of sets $\mathscr{S}$ is *linked* if $A \cap B \neq \emptyset$ for all $A, B \in \mathscr{S}$. The set of all maximal linked families of closed subsets of Y is denoted by λY. If G is an open subset of

Y, we define

$$G^+ = \{m \in \lambda Y\text{: there exists } F \in m \text{ with } F \subseteq G\}.$$

λY is endowed with a topology by taking all sets of the form G^+, for G open in Y, as an open subbase. With this topology, λY is referred to as the superextension of Y. Superextensions were introduced by J. De Groot in [4] and our basic references for them are [9] and [18].

In the construction of our example, we assume familiarity with the basic notions of forcing. We use standard terminology and the reader is referred to [5] for elaboration.

3. Exp Y can be an S-space

In [7], Malyhin proved that if exp Y has countable spread, then Y is compact, hereditarily separable and hereditarily Lindelöf. He asked whether Y must in fact be metrizable. In [3], it is shown that MA + $\neg$CH implies that this is the case. We now show that this need not be the case, i.e., Malyhin's question is independent of ZFC.

If exp Y is hereditarily Lindelöf, then $\langle Y\rangle^1$ is a G_δ in $\langle Y\rangle^2$. This implies that Y has a G_δ-diagonal. Since Y is compact, by a result in [11], Y is metrizable.

3.1. Theorem. *It is consistent with* ZFC *that there exists a compact* 0*-dimensional Hausdorff space X with the following properties*:

(1) exp X *has countable spread but X is not metrizable, moreover* exp X *is hereditarily separable, hence X^2 is a strong S-space, while X is not an S-space.*

(2) *X has no uncountable weakly separated subspaces.*

The proof will be completed by some preliminaries and a series of lemmas.

Let V be our universe of sets. We work in V now. If F is a finite subset of ω_1 and $\mathcal{F} \subseteq 2^F$, then we say that *$\mathcal{F}$ splits F* if for each $\alpha \in F$, there exists $s, t \in \mathcal{F}$ such that

$$s \restriction \{\beta \in F\colon \beta < \alpha\} = t \restriction \{\beta \in F\colon \beta < \alpha\} \quad \text{and} \quad s(\alpha) \neq t(\alpha).$$

Define $P = \{(F, \mathcal{F})\colon F$ is a finite subset of ω_1, $\mathcal{F} \subseteq 2^F$, and $\mathcal{F}$ splits $F\}$. A partial order $\leqslant$ is defined on P as follows: $(S, \mathcal{S}) \leqslant (T, \mathcal{T})$ if $T \subseteq S$, for each $s \in \mathcal{S}$, $s \restriction T \in \mathcal{T}$, and for each $t \in \mathcal{T}$ there exists $s \in \mathcal{S}$ such that $t \subseteq s$. A proof that $(P, \leqslant)$ satisfies the countable chain condition appears in Lemma 3.3. It is easily checked that for each finite subset H of ω_1, $\{(F, \mathcal{F})\colon H \subseteq F\}$ is a dense subset of P.

Let G be a generic filter on P. Since P has the countable chain condition, $V[G]$ has the same cardinals as V.

We work now in $V[G]$. Define $H(\omega_1)=\{s: s$ is a finite partial function from ω_1 to $2\}$. For each $s\in H(\omega_1)$, define

$$[s]=\{f\in 2^{\omega_1}: s\subseteq f\}.$$

Let us define a closed subspace X of 2^{ω_1} by $f\in X$ iff for each $(F,\mathscr{F})\in G$, $f\restriction F\in\mathscr{F}$. Then,

$$X=\bigcap_{(F,\mathscr{F})\in G}\Big(\bigcup_{s\in\mathscr{F}}[s]\Big).$$

X, with the subspace topology, is our example; being a closed subspace of 2^{ω_1}, X is a compact, 0-dimensional, Hausdorff space. Note that if $(F,\mathscr{F})\in G$ and $t\in\mathscr{F}$, then $[t]\cap X\neq\emptyset$. This follows from the fact that since G is a filter, $\{[t]\}\cup\{\bigcup_{s\in\mathscr{S}}[s]: (S,\mathscr{S})\in G\}$ is a centered collection of closed subsets of 2^{ω_1} and hence has non-empty intersection.

3.2. Lemma. *In $V[G]$, X is not metrizable.*

Proof. Since X is compact and 0-dimensional, we need only show that X contains uncountably many distinct clopen subsets. In fact, $\alpha<\beta$ implies $[(\alpha,0)]\cap X\neq[(\beta,0)]\cap X$. For, $G\cap\{(F,\mathscr{F}): \{\alpha,\beta\}\subseteq F\}\neq\emptyset$. If $(F,\mathscr{F})$ is in this intersection, then, as $\mathscr{F}$ splits F, there exist $s,t\in\mathscr{F}$ such that $s(\alpha)=t(\alpha)$ and $s(\beta)\neq t(\beta)$. $(F,\mathscr{F})\in G$ implies there exist $f\in[s]\cap X$ and $g\in[t]\cap X$. Then, at least one of f and g is in the symmetric difference of $[(\alpha,0)]\cap X$ and $[(\beta,0)]\cap X$.

3.3. Lemma. *In $V[G]$, X has the following property,*

(*) *For each $n<\omega$ and for each $\{s_\alpha^i: i<n, \alpha<\omega_1\}\subseteq H(\omega_1)$, there exist $\alpha<\omega_1$ and a finite subset K of ω_1 with $\alpha<K$ such that:*

(a) $$\bigcup_{i<n}([s_\alpha^i]\cap X)\subseteq\bigcap_{\beta\in K}\Big(\bigcup_{i<n}[s_\beta^i]\Big)$$

and

(b) $$\prod_{i<n}([s_\alpha^i]\cap X)\subseteq\bigcup_{\beta\in K}\Big(\prod_{i<n}[s_\beta^i]\Big).$$

Proof. Let $(F,\mathscr{F})\in P$ force our hypothesis. We will find a stronger condition $(S,\mathscr{S})\in P$ that will force our conclusion. For each $\alpha\in\omega_1$, choose $(F_\alpha,\mathscr{F}_\alpha)\in P$ such that:

(i) $(F_\alpha,\mathscr{F}_\alpha)\leq(F,\mathscr{F})$.

(ii) $\bigcup_{i<n}\operatorname{dom}(s_\alpha^i)\subseteq F_\alpha$.

By a counting argument, there exist an uncountable subset A of ω_1 and $m,p<\omega$ such that:

(iii) For each $\alpha \in A$, $|F_\alpha| = m$ and $|\mathcal{F}_\alpha| = p$.

(iv) $\{F_\alpha : \alpha \in A\}$ is a delta-system with root R.

(v) $\alpha, \beta \in A$, $\alpha < \beta$ implies $R < F_\alpha - R < F_\beta - R$.

For each $\alpha \in A$, let $F_\alpha = \{\alpha_j : j < m\}$ where $i < j$ implies $\alpha_i < \alpha_j$. Let $\mathcal{F}_\alpha = \{t_\alpha^k : k < p\}$. Since there are only finitely many p-sequences of functions from $\{1, \ldots, m\}$ into $\{0, 1\}$ and only finitely many n-sequences of partial functions $\{1, \ldots, m\}$ into $\{0, 1\}$, we may further assume that:

(vi) For each $\alpha, \beta \in A$, for each $j < m$, for each $k < p$, $t_\alpha^k(\alpha_j) = t_\beta^k(\beta_j)$ (in particular, $t_\alpha^k \restriction R = t_\beta^k \restriction R$).

(vii) For each $\alpha, \beta \in A$, for each $i < n$, for each $k < p$, $s_\alpha^i \subseteq t_\alpha^k$ iff $s_\beta^i \subseteq t_\beta^k$.

Now, choose $\alpha \in A$ and a subset K of A such that $\alpha < K$ and $|K| = n + 1$. Let $S = F_\alpha \cup \bigcup_{\beta \in K} F_\beta$. Define

$$\mathcal{S}_1 = \left\{ t_\alpha^k \cup \bigcup_{\beta \in K} t_\beta^k : k < p \right\}.$$

For each $j < m$, split α_j by $t_\alpha^{a_j}$ and $t_\alpha^{b_j}$ of $\mathcal{F}_\alpha$; then interchanging if necessary, assume that if there exists $i < n$ with $s_\alpha^i \subseteq t_\alpha^{a_j}$, then there exists $l < n$ with $s_\alpha^l \subseteq t_\alpha^{b_j}$. Define

$$\mathcal{S}_2 = \left\{ t_\alpha^{a_j} \cup \bigcup_{\delta \in K - \{\beta\}} t_\delta^{a_j} \cup t_\beta^{b_j} : \beta \in K \text{ and } j < m \right\}.$$

Define $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$. Then, $(S, \mathcal{S}) \in P$ and $(S, \mathcal{S}) \leqslant (F, \mathcal{F})$.

Note that applying the reductions (iii) to (vi) to *any* uncountable $\{(F_\alpha, \mathcal{F}_\alpha) : \alpha < \omega_1\} \subseteq P$, we see that for any $\alpha, \beta \in A$, $(S, \mathcal{S}) \leqslant (F_\alpha, \mathcal{F}_\alpha)$ and $(S, \mathcal{S}) \leqslant (F_\beta, \mathcal{F}_\beta)$. This shows that P has the countable chain condition.

To show that $(S, \mathcal{S})$ forces (a), assume that $(S, \mathcal{S}) \in G$, $f \in \bigcup_{i<n} ([s_\alpha^i] \cap X)$ and $\beta \in K$. Since $f \in X$ and $(S, \mathcal{S}) \in G$, we have $f \restriction S \in \mathcal{S}$. But all functions $t \in \mathcal{S}$ have the property that if there exists $i < n$ such that $s_\alpha^i \subseteq t$, then there exists $l < n$ such that $s_\beta^l \subseteq t$ (recall conditions (vi) and (vii) and our definitions of $\mathcal{S}_1$ and $\mathcal{S}_2$). Thus, there exists $l < n$ such that $s_\beta^l \subseteq f \restriction S$ and consequently, $f \in [s_\beta^l] \cap X$.

To show that $(S, \mathcal{S})$ forces (b), assume that $(S, \mathcal{S}) \in G$ and that for each $i < n$, $f_i \in [s_\alpha^i] \cap X$. For each $i < n$, we have that $f_i \restriction S \in \mathcal{S}$. Since there are $n+1$ β's in K and only n f_i's, there exists $\beta \in K$ such that for each $i < n$ and for each $j < m$, $f_i(\alpha_j) = f_i(\beta_j)$. Since for each $i < n$, $\mathrm{dom}(s_\alpha^i) \subseteq F_\alpha$ and $\mathrm{dom}(s_\beta^i) \subseteq F_\beta$, we have that for each $i < n$, $f_i \in [s_\beta^i] \cap X$.

3.4. Lemma. *If a subspace X of 2^{ω_1} has the property* $(*)$, *then* $\exp X$ *is hereditarily separable.*

Proof. Assume not. Then, there exists an uncountable left-separated subspace $\{C_\alpha : \alpha < \omega_1\}$ of $\exp X$. By passing to an uncountable subset of ω_1, we may assume that there exists an $n < \omega$ and for each $\alpha < \omega_1$ an open set $\langle [s_\alpha^0] \cap X, \ldots, [s_\alpha^{n-1}] \cap X \rangle$ such that for each $\alpha < \omega_1$, $C_\alpha \in \langle [s_\alpha^0] \cap X, \ldots, [s_\alpha^{n-1}] \cap X \rangle$ and

$$\{C_\beta : \beta < \alpha\} \cap \langle [s_\alpha^0] \cap X, \ldots, [s_\alpha^{n-1}] \cap X \rangle = \emptyset.$$

Consider $\{s^i_\alpha: i<n, \alpha<\omega_1\}\subseteq H(\omega_1)$. Let $\alpha<K$ witness property (*). For each $i<n$, choose $f_i\in C_\alpha\cap[s^i_\alpha]$. By (b), there exists $\beta\in K$ such that if $i<n$, $f_i\in[s^i_\beta]$. By (a), we have

$$C_\alpha\subseteq\bigcup_{i<n}([s^i_\alpha]\cap X)\subseteq\bigcup_{i<n}[s^i_\beta].$$

This implies that $C_\alpha\in\langle[s^0_\beta]\cap X,\ldots,[s^{n-1}_\beta]\cap X\rangle$, a contradiction. Hence, our lemma is proved.

3.5. Lemma. *In $V[G]$, X has the following property,*

(**) *If $\{s_\alpha:\alpha<\omega_1\}\subseteq H(\omega_1)$, then there exists $\alpha\neq\beta\neq\gamma\neq\alpha$ in ω_1 such that $[s_\alpha]\cap X=([s_\beta]\cap X)\cup([s_\gamma]\cap X)$.*

Proof. Let $(F,\mathcal{F})\in P$ force our hypothesis. We will find a stronger condition $(S,\mathcal{S})\in P$ that will force our conclusion. Using the same notation and conventions of Lemma 3.3, but with $n=1$ and $s^0_\alpha=s_\alpha$, choose an uncountable subset A of ω_1 and for each $\alpha\in A$ an $(F,\mathcal{F})\in P$ satisfying (i) to (vii) of said lemma.

Now, choose $\alpha<\beta<\gamma$ in A. Let $S=F_\alpha\cup F_\beta\cup F_\gamma$. Define

$$\mathcal{S}_1=\{t^k_\alpha\cup t^k_\beta\cup t^k_\gamma: k<p\}.$$

For each $j<m$, split a_j by $t^{a_j}_\alpha$ and $t^{b_j}_\alpha$ of $\mathcal{F}_\alpha$; then interchanging if necessary, assume that if $s_\alpha\subseteq t^{b_j}_\alpha$, then $s_\alpha\subseteq t^{a_j}_\alpha$. Define

$$\mathcal{S}_2=\{t^{a_j}_\alpha\cup t^{b_j}_\beta\cup t^{a_j}_\gamma: j<m\}\cup\{t^{a_j}_\alpha\cup t^{a_j}_\beta\cup t^{b_j}_\gamma: j<m\}.$$

Define $\mathcal{S}=\mathcal{S}_1\cup\mathcal{S}_2$. Then $(S,\mathcal{S})\in P$ and $(S,\mathcal{S})\leqslant(F,\mathcal{F})$.

To show that $(S,\mathcal{S})$ forces $[s_\alpha]\cap X=([s_\beta]\cap X)\cup([s_\gamma]\cap X)$, just note that no finite function of $\mathcal{S}$ can serve as a counterexample to this equation.

Remark. Note that the proof of Lemma 3.5 can be adjusted to show that X, in $V[G]$, has the following stronger property:

(***) If $A\subseteq\mathcal{B}(X)$ is uncountable, then there exists $a\neq b\neq c\neq a$ in A such that $a=b\cup c$.

It is easy to see that (***) implies that $\mathcal{B}(X)$ has no uncountable chains nor antichains. It is an unpublished result of S. Shelah that (***) implies $|\mathcal{B}(X)|\leqslant\omega_1$. Shelah's argument can be used to show that already (**) implies $|\mathcal{B}(X)|\leqslant\omega_1$, more precisely: If a compact 0-dimensional Hausdorff space X has a basis $\mathcal{B}\subseteq\mathcal{B}(X)$ with the property (**), then $|\mathcal{B}(X)|\leqslant\omega_1$. We must mention that it was already known to be consistent that uncountable Boolean algebras with no uncountable chains or antichains exist, cf. [1].

3.6. Lemma. *In $V[G]$, X has no uncountable weakly separated subspaces.*

Proof. Assume not. Then, there exists an uncountable $Y \subseteq X$ and a reflexive and antisymmetric relation R on Y such that for each $y \in Y$, $\{x \in Y: xRy\}$ is open in Y. For each $y \in Y$, define $O_y = \{x \in Y: xRy\}$. For each $y \in Y$, let V_y be an open set of X such that $O_y = V_y \cap Y$. Since R is reflexive, $y \in O_y$ for each $y \in Y$. For each $y \in Y$, choose $s_y \in H(\omega_1)$ such that $y \in [s_y] \cap X \subseteq V_y$. Apply Lemma 3.5 to the uncountable set $\{s_y: y \in Y\}$ to get three distinct elements x, y, z of Y such that $[s_x] \cap X = ([s_y] \cap X) \cup ([s_z] \cap X)$. Since $x \in [s_x] \cap X$, we have $x \in [s_y] \cap X$ or $x \in [s_z] \cap X$. Without loss of generality, assume $x \in [s_y] \cap X$. Since $y \in [s_y] \cap X$, we have $y \in [s_x] \cap X$. But, then $x \in V_y \cap Y = O_y$ and $y \in V_x \cap Y = O_x$ and this contradicts antisymmetry of R. Hence, our lemma is proved.

Remark. M. Tkacenko [14] had previously constructed from CH, a compact non-metrizable space Z such that Z has no uncountable weakly separated subspaces. It is unknown whether such an example exists in ZFC.

4. More on exp Y

The theorems and propositions of this section and Section 5 do not require any axioms beyond ZFC.

If exp Y is hereditarily separable, then exp Y has countable spread. We prove now that the converse is true.

4.1. Theorem. *If* exp Y *has countable spread, then* exp Y *is hereditarily separable.*

Proof. Let exp Y have countable spread. Assume that exp Y is not hereditarily separable. Let $\{C_\alpha: \alpha < \omega_1\}$ be an uncountable left-separated subspace of exp Y. Let $W = \bigcup\{O: O$ is open second countable subspace of $Y\}$. Since Y is hereditarily Lindelöf, W has a countable basis. Define $F = Y - W$. F is a closed subspace of Y. Since Y is hereditarily separable, choose a countable dense subset D of F. If there exists $d \in D$ such that $|\{\alpha < \omega_1: d \notin C_\alpha\}| = \omega_1$, then choose open sets U, V of Y such that $d \in U \subseteq \mathrm{cl}_Y U \subseteq V$ and $V \cap C_\alpha = \emptyset$ for uncountably many $\alpha < \omega_1$ (remember that Y is first countable). Since $d \in F$, U does not have a countable basis and thus $\exp(\mathrm{cl}_Y U)$ is not hereditarily Lindelöf. Also, $\exp(Y - V)$ is not hereditarily separable since it contains uncountably many C_α's. Hence, by a result in [19], $\exp(Y - V) \times \exp(\mathrm{cl}_Y U)$ has uncountable spread. But, $\exp(Y - V) \times \exp(\mathrm{cl}_Y U)$ is embedded in exp Y. This contradicts countable spread of exp Y. Consequently, by passing to a co-countable subset of ω_1, we may assume that $D \subseteq \bigcap_{\alpha<\omega_1} C_\alpha$, which implies that $F \subseteq \bigcap_{\alpha<\omega_1} C_\alpha$. Let $C_\alpha \in \langle O_\alpha^1, \ldots, O_\alpha^{m_\alpha}, \ldots, O_\alpha^{n_\alpha}\rangle$ where $O_\alpha^i \cap F \neq \emptyset$ iff $i \leq m_\alpha$. Plus, $\{C_\beta: \beta < \alpha\} \cap \langle O_\alpha^1, \ldots, O_\alpha^{n_\alpha}\rangle = \emptyset$.

Consider the quotient space Y/F of Y by collapsing F to a point. Let π be the quotient map. Since Y is compact, W is second countable and F is a G_δ in Y, we have that Y/F is compact metrizable. By a theorem of F. Hausdorff, $\exp(Y/F)$ is compact metrizable and hence hereditarily separable. But, $\{\pi(C_\alpha): \alpha < \omega_1\}$ is an uncountable left-separated subspace of $\exp(Y/F)$ as the open sets

$$\langle \pi(\bigcup\{O^i_\alpha : i \leq m_\alpha\}), \pi(O^{m_\alpha+1}_\alpha), \ldots, \pi(O^{n_\alpha}_\alpha)\rangle$$

will testify. This contradiction implies that $\exp Y$ must be hereditarily separable.

Remark. One can see from this theorem, why MA + $\neg$CH implies that if $\exp Y$ has countable spread, then Y is compact metrizable. For, under MA + $\neg$CH, there are no compact S-spaces, see [12].

The following proposition may be known but we include a proof for completeness sake.

4.2. Proposition. *For each $n < \omega$, Y^n has countable spread iff $\langle Y\rangle^n$ has countable spread.*

Proof. The mapping $h_n : Y^n \to \langle Y\rangle^n$ given by $h((x_1, \ldots, x_n)) = \{x_1, \ldots, x_n\}$ is a continuous onto map; it is also closed, but open only for $n = 1$ and $n = 2$. Thus, if Y^n has countable spread, then so does $\langle Y\rangle^n$.

Let $\langle Y\rangle^n$ have countable spread. Assume that Y^n does not and let D be an uncountable discrete subspace of Y^n. Let S_n denote the group of all permutations of $\{1, \ldots, n\}$. For each $\sigma \in S_n$, for each $x = (x_1, \ldots, x_n) \in Y^n$ and for each $A \subseteq Y^n$, define $\sigma(x) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ and $\sigma(A) = \{\sigma(x): x \in A\}$.

Henceforth, cl will denote closure in Y^n.

We claim that there exists an uncountable $A \subseteq D$ such that for each $x \in A$,

$$\{\sigma(x): \sigma \in S_n\} \cap \mathrm{cl}(A - \{x\}) = \emptyset.$$

Then $\{\{x_1, \ldots, x_n\}: (x_1, \ldots, x_n) \in A\}$ is an uncountable discrete subspace of $\langle Y\rangle^n$. Contradiction. To prove this claim, it suffices to show that for a fixed $\sigma \in S_n$, there exists an uncountable $A_\sigma \subseteq D$ such that for each $x \in A_\sigma$, $\sigma(x) \notin \mathrm{cl}(A_\sigma - \{x\})$. To this end, let $\sigma \in S_n$ and let the order of σ be k. Since D is discrete in Y^n, $k \geq 2$. Striving for a contradiction, assume that for each uncountable $A \subseteq D$, there exists an $x \in A$ such that $\sigma(x) \in \mathrm{cl}(A - \{x\})$. Inductively construct disjoint uncountable subsets of D, $A_0, \ldots, A_k$, such that for each i, $1 \leq i \leq k$, $\sigma(A) \subseteq \mathrm{cl}\, A_{i-1}$. But then, $\sigma(A_k) \subseteq \mathrm{cl}\, A_{k-1}$ and thus $\sigma^2(A_k) \subseteq \mathrm{cl}\, \sigma(A_{k-1}) \subseteq \mathrm{cl}\, A_{k-2}$. Applying σ $k-2$ more times, we get $A_k = \sigma^k(A_k) \subseteq \mathrm{cl}\, A_0$. This contradicts D being discrete.

Remark 1. Consider our example X of Section 3. The preceding proposition implies that X^n has countable spread of all n. By a result in [19], we have that either X^n is hereditarily separable for all n or X^n is hereditarily Lindelöf for all n. Since X^2 is not hereditarily Lindelöf, we see that X^2 is a strong S-space (every finite power is an S-space) but X is not, it is hereditarily Lindelöf.

Remark 2. An alternate proof that MA + $\neg$CH implies that if exp Y has countable spread, then Y is compact and metric goes as follows: If exp Y has countable spread, then $\langle Y\rangle^n$ has countable spread for all n; hence, by Proposition 4.2, Y^n has countable spread for all n. Hence, either Y^n is hereditarily separable for all n or Y^n is hereditarily Lindelöf for all n. Kunen [6] has shown, under MA + $\neg$CH, that Y^n is hereditarily separable for all n iff Y^n is hereditarily Lindelöf for all n. Hence, Y^2 is hereditarily Lindelöf. Since Y is compact, Y is metrizable.

Remark 3. Is Proposition 4.2 a special case of a more general result? It is not true that if Z is a continuous and $\leqslant n$ to 1 image of a compact space Y and if Z has countable spread, then Y has countable spread. Consider the natural projection map of the Alexandroff double of the closed unit interval onto the closed unit interval. In our proposition, when $n = 2$, the mapping involved is also open. Ortwin Förster (private communication) has shown that if Z is a continuous, open and $\leqslant 2$ to 1 image of a space Y, then Y has countable spread if Z has countable spread. We challenge the reader to come up with a proof of this fact. Of course, in Proposition 4.2, the mapping involved is not open for $n \geqslant 3$. This necessitated using the specific nature of the spaces involved.

5. Some things on λY

A. Verbeek [18] has proven that λY is compact and metrizable iff Y is compact and metrizable. E. Van Douwen, see [9], has proven that if λY is hereditarily Lindelöf, then Y is compact and metrizable. Assuming MA + $\neg$CH, hereditarily Lindelöf can be replaced by countable spread in the above, see [3]. We now have the following result for 0-dimensional spaces.

5.1. Theorem. *Let Y be 0-dimensional. Then λY has countable spread iff Y is compact and metrizable.*

Proof. Let λY have countable spread. Then, by a result in [3], exp Y has countable spread. Now, by Malyhin's result [7], Y is compact. Assume that Y is not metrizable. Then, Y is not second countable. Choose a clopen set U of Y such that U contains uncountably many clopen sets and $1 < |Y - U|$. Decompose $Y - U = A \cup B$ where A and B are disjoint non-empty clopen sets of Y.

Inductively define $\{A_\alpha : \alpha < \omega_1\} \subseteq \mathscr{B}(Y)$ such that for each $\alpha < \omega_1$, A_α is a non-empty subset of U and $\beta < \alpha$ implies $A_\alpha \notin \{A_\beta, U - A_\beta\}$.

Claim. $\{(A \cup A_\alpha)^+ \cap (A \cup (U - A_\alpha))^+ : \alpha < \omega_1\}$ is a strong antichain in $\mathscr{B}(\lambda Y)$. Hence, λY has uncountable spread. Contradiction.

Proof of claim. Let F be a finite subset of ω_1 and let $\alpha \notin F$. If there exists $\beta \in F$ such that $[A_\alpha \subseteq A_\beta$ or $U - A_\alpha \subseteq A_\beta]$ and $[A_\alpha \subseteq U - A_\beta$ or $U - A_\alpha \subseteq U - A_\beta]$, then $A_\alpha = A_\beta$ or $A_\alpha = U - A_\beta$. Thus, for each $\beta \in F$ we can choose $h(\beta) \in \{A_\beta, U - A_\beta\}$

such that $A_\alpha \not\subseteq h(\beta)$ and $U - A_\alpha \not\subseteq h(\beta)$. Then

$$\{A \cup A_\alpha, A \cup (U - A_\alpha)\} \cup \{Y - (A \cup h(\beta)) : \beta \in F\}$$

is a linked collection. Extend this collection to an $m \in \lambda Y$. Then,

$$m \in [(A \cup A_\alpha)^+ \cap (A \cup (U - A_\alpha))^+] - \bigcup_{\beta \in F} [(A \cup A_\beta)^+ \cap (A \cup (U - A_\beta))^+].$$

Hence, Y is metrizable.

Question. Is Theorem 5.1 true for arbitrary Y?

Our next result relates to the problem "When is λY first countable?" J. van Mill [9] has proven that λY is first countable iff λY has countable tightness. M. van de Vel has proven that if λY is first countable, then Y has the shrinking property (for a definition, see [15]). E. van Douwen, see [9], has proven that if λY is first countable, then Y is compact, hereditarily separable and hereditarily Lindelöf. Must Y be metric in ZFC? We shall see that the answer is no, (E. Wattel [16] has shown that the Alexandroff double arrow line has a non-first countable superextension). It is interesting to note that exp Y is first countable iff Y is compact, hereditarily separable, and hereditarily Lindelöf, see [17].

5.2. Proposition. *If* exp Y *has countable spread, then* λY *is first countable.*

Proof. Let exp Y have countable spread. Assume that λY is not first countable. Then, there exists $m \in \lambda Y$ such that $\{m\}$ is not a G_δ. We will inductively define $\{C_\alpha : \alpha < \omega_1\} \subseteq m$ and open sets $\{O_\alpha : \alpha < \omega_1\}$ of Y such that $C_\alpha \subseteq O_\alpha$ and $\beta < \alpha$ implies $C_\beta \not\subseteq O_\alpha$. Then, for each $\alpha < \omega_1$, $C_\alpha \in \langle O_\alpha \rangle$ and this shows that $\{C_\alpha : \alpha < \omega_1\}$ is an uncountable left-separated subspace of exp Y, contrary to Theorem 4.1.

Assume that we have so defined our sets C_β and O_β, for $\beta < \alpha$. Since Y is perfectly normal, for each $\beta < \alpha$, $C_\beta = \bigcup_{i<\omega} O_\beta^i$ where the O_β^i's are a sequence of open sets of Y with cl $O_\beta^{i+1} \subseteq O_\beta^i$ for each $i < \omega$. Now,

$$m \in \cap \{(O_\beta^i)^+ : \beta < \alpha,\ i < \omega\} = \{m' : \text{for each } \beta < \alpha,\ C_\beta \in m'\}$$

by compactness of Y. Since $\{m\}$ is not a G_δ, choose $m' \in \{m' :$ for each $\beta < \alpha$, $C_\beta \in m'\} - \{m\}$. Choose $C_\alpha \in m$ such that $C_\alpha \notin m'$. Therefore, there exists $C \in m'$ with $C \cap C_\alpha = \emptyset$. Define $O_\alpha = Y - C$. Then, if $\beta < \alpha$, then $\{C_\beta, C\} \subseteq m'$ and so $C_\beta \cap C \neq \emptyset$ and hence $C_\beta \not\subseteq O_\alpha$.

Remark. Our example X of Section 3 (using Proposition 5.2) serves as a consistent example of a 0-dimensional non-metric space whose superextension is first countable. Such a space cannot exist in ZFC alone. This follows from the following theorem and the fact that

$$\mathrm{ZFC} + 2^{\omega} < 2^{\omega_1} + (\mathrm{B})$$

is consistent, where (B) denotes the following statement:

(B) Every uncountable Boolean algebra has an uncountable antichain.

This consistency result can be proved using an idea of S. Shelah that he used to prove Theorem 4 of [10] no new ideas are needed.

5.3. Theorem. *Assume* (B). *If* Y *is* 0-*dimensional and* $|\lambda Y| < 2^{\omega_1}$, *then* Y *is compact metrizable.*

Proof. Let Y be 0-dimensional and $|\lambda Y| < 2^{\omega_1}$. Assume that Y does not have a countable basis. Choose three mutually disjoint non-empty clopen subsets A, B and C of Y such that $\mathscr{B}(C)$ is uncountable. We conclude that $\mathscr{B}(C)$ contains an uncountable antichain $\{C_\alpha : \alpha < \omega_1\}$. For each $\alpha < \omega_1$, set $D^0_\alpha = A \cup C_\alpha$ and $D^1_\alpha = B \cup (C - C_\alpha)$. For each $f \in 2^{\omega_1}$, set $L_f = \{D^{f(\alpha)}_\alpha : \alpha < \omega_1\}$. Each L_f is a linked collection of closed subsets of Y and thus can be extended to a maximal linked family $m_f \supseteq L_f$. Distinct f's give rise to distinct m_f's. Hence, $|\lambda Y| \geq 2^{\omega_1}$. This is a contradiction. Consequently, Y has a countable basis.

Furthermore, Y must be countably compact; because if Y has an infinite closed discrete subspace, the $|\lambda Y| \geq 2^{2^{\omega}}$. Thus, Y is compact metrizable.

5.4. Corollary. *Assume* $2^{\omega} < 2^{\omega_1} + (\mathrm{B})$. *If* Y *is* 0-*dimensional and* λY *is first countable, then* Y *is compact metrizable.*

Proof. Let Y be 0-dimensional and λY be first countable. By A. Arhangelskii's result [0] that first countable compact spaces have cardinality at most 2^{ω}, we conclude that $|\lambda Y| < 2^{\omega_1}$ and we now invoke the preceding theorem.

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